Efficient Representation of Realistic 3D Static Magnetic Fields for Symplectic Tracking

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The Problem

Long-term tracking of charged particles is a fundamental problem of accelerator physics, plasma physics, etc. Static magnetic and electric fields must obey Maxwell's equations everywhere in the domain of interest. Failing to satisfy this condition leads to spurious sources, which in turn leads to an energy drift during the tracking and non-conservation of phase space volume. All 3D grid methods suffer from this problem, because there is no known interpolation satisfying the Maxwell's equations while being continuous between the 3D grid cells.
The second requirement is the continuity of the potentials. Cutting the potentials at some distance, even far from a magnetic element, leads to a systematic error which accumulates during the tracking causing an energy drift and non-conservation of phase space volume.

The Solution

The magnetic or electric fields are expressed in terms of point sources on a surface surrounding each magnet. These sources are placed outside of the boundary surface at some distance, and their strength is set such that they reproduce the magnetic or electric field at the surface by solving a system of linear equations. We do this for each type of magnet, then the accelerator is assembled as a collection of sources shifted and rotated at the right place.
After the field of the entire machine is reproduced as a sum of point source fields, it can be evaluated anywhere inside the beam region. The field obtained from the sources is continuous and satisfy the Maxwell equations to machine precision. However, tracking particles directly with this method is too slow to be practical.
Several orders of magnitude improvement can be achieved by using a local description of the potentials. Regular solid harmonics are the canonical representation for harmonic functions inside a sphere. A key characteristic of the algorithm is the description of vector and scalar potentials by solid harmonics inside a set of overlapping spheres covering the volume of interest.
The potentials satisfy exactly Maxwell's equation inside the spheres. The discontinuity between the spheres decrease exponentially with the degree of solid harmonics expansion and can be easily kept close to machine precision. The representation of the potentials in terms of solid harmonics is optimal in terms of memory and allows fast evaluation. We recommend reading the papers [1, 2] to understand the algorithm in detail. Our software called SIMPA [3] implements this algorithm and it was used to calculate frequency analysis and dynamic apertures of the ELENA machine.

References

[1] Lajos Bojtár: *Efficient evaluation of arbitrary static electromagnetic fields with applications for symplectic particle tracking*, Nuclear Instruments and Methods A,948:162841, 2019.

[2] Lajos Bojtár: *Frequency analysis and dynamic aperture studies in a low energy antiproton ring with realistic 3D magnetic fields*, Phys. Rev. Accel. Beams 23:104002,Oct 2020

[3] https://simpa-project.web.cern.ch/

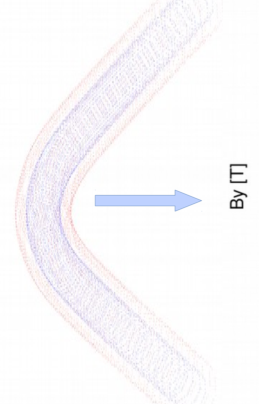
[4] https://github.com/nkarast/PyNAFF

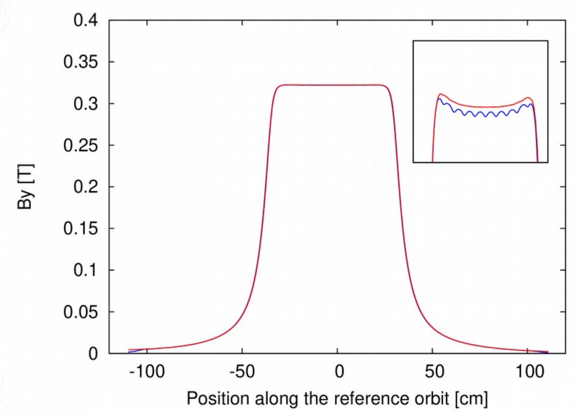
Field map preparation workflow

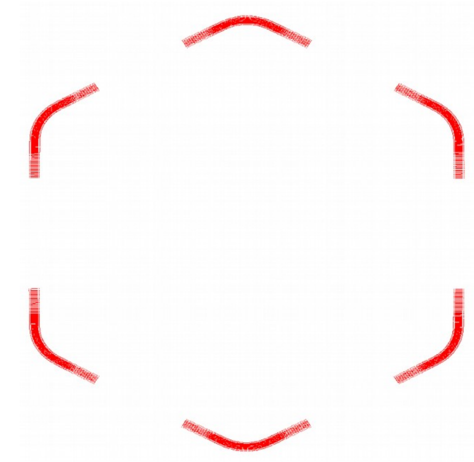
The input magnetic or electric field from CAD software or measurements reproduced on a surface by point sources. A system of linear equation is solved to calculate the strengths of the sources. The field of the sources is analytic and satisfy the Maxwell equations exactly.

An arrangement of point sources is rotated and shifted to the correct place for each magnet to construct a machine.

Input field data given on a boundary surface surrounding a magnet







Now have a continuous analytic field for the machine and could track particles. Would be very slow, as the contribution of each source must be summed at each time step.

Using solid harmonic expansion of the vector potential a local representation of the field inside a ball is given. This is much faster to evaluate.

$$f(r,\theta,\phi) = \frac{1}{R}\sum_{\ell=0}^{\ell_{max}}\sum_{m=-\ell}^{\ell} r^{\ell} c_{\ell m}\tilde{P}_{\ell}^{m}(\cos\theta)\,\Phi(\phi;m)$$

$$\Phi(\phi;m)=\begin{cases}\sqrt{2}\sin(|m|\phi) & \text{if } m<0\\ 1 & \text{if } m=0\\ \sqrt{2}\cos(m\phi) & \text{if } m>0\end{cases}.$$

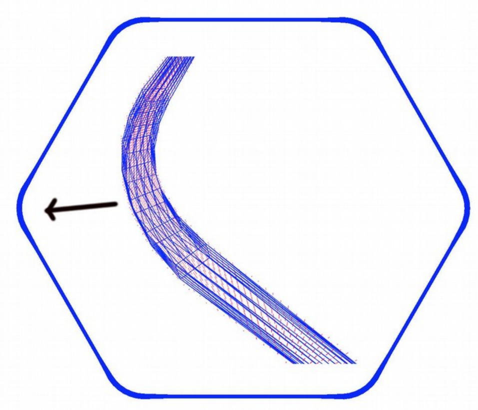
The beam region Must be covered by overlapping balls.

Two options are implemented:

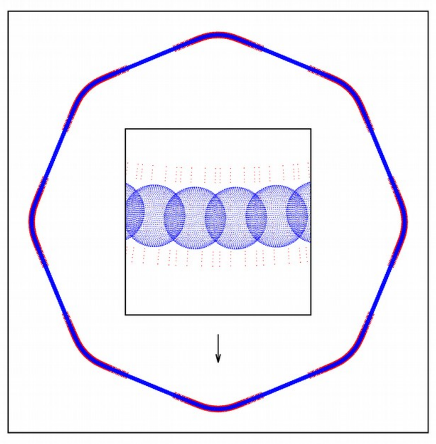
- Single row of balls around the orbit

-Small balls on a HCP lattice filling an arbitrary volume

Smaller overlapping balls placed on HCP lattice can cover general geometries and faster to evaluate, but takes longer to prepare the field map.



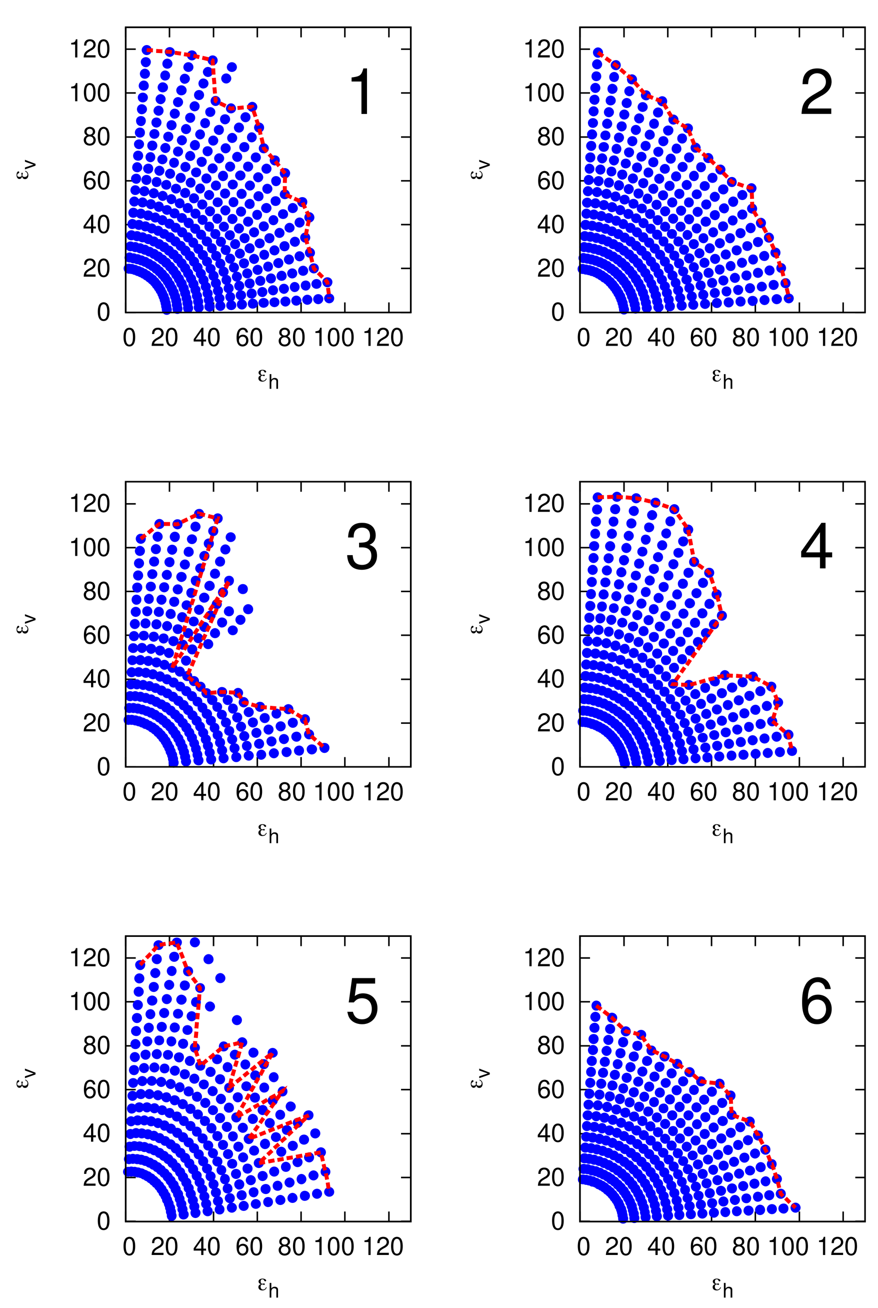
Single row of balls on the orbit. Faster to prepare the field map, but slower at evaluation and less general.



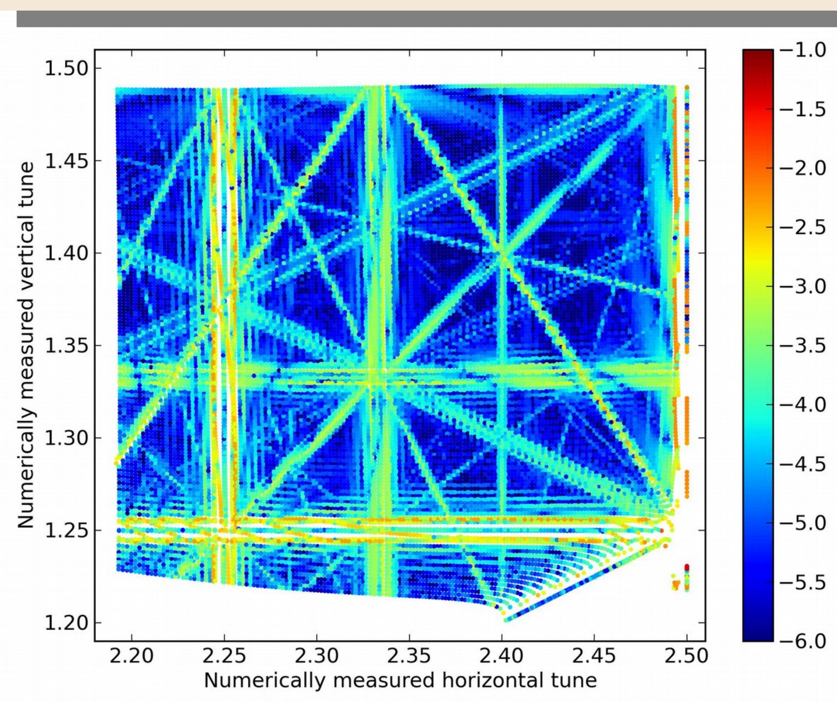
Frequency Analysis and Dynamic Aperture Results in ELENA

FREQUENCY ANALYSIS: The tunes were scanned in 160 steps in both directions giving 25600 initial conditions. Each particle was tracked for 300 turns and the phase space variables were saved at a single longitudinal position for each turn. To calculate the tune drift $D = log_{10}(\sqrt{\Delta Q_h^2 + \Delta Q_v^2})$, we split the 300 turns into two sets and calculated $\Delta Q_{h,v}$ with PyNAFF [4]. Then plotted D as colors for each tunes.
DYNAMIC APERTURE: The dynamic aperture of six working points has been calculated to compare resonance conditions with a non-resonant case. The selected set of tunes is indicated on the right bottom figure. For each of the 6 tune settings particles with a range of initial $\epsilon_{h,v}$ were tracked. The particles surviving at leas 10^4 turns are displayed on left figure. For details of the procedure see [2].

Dynamic apertures at the numbered tune points. The blue dots correspond to initial conditions of particles which survived at least 1E4 turns.



Frequency analysis results in ELENA without the electron cooler.



Frequency analysis results in ELENA with electron cooler magnets included.